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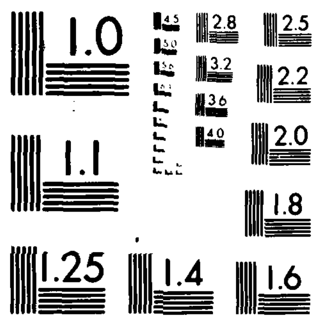
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A DECOMPOSITION PROCEDURE FOR LARGE SCALE OPTIMAL PLASTIC DESIGN PROBLEMS

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ABSTRACT

A decomposition procedure is proposed in this paper for solving a class of large scale optimal design problems for perfectly plastic structures under several alternative loading conditions. The conventional finite element method is used to cast the problem into a finite dimensional constrained nonlinear programming problem. Structures of practically meaningful size and complexity tend to give rise to a large number of variables and constraints in the corresponding mathematical model. The difficulty is that the state-of-the-art Mathematical Programming theory does not provide reliable and efficient ways of solving large scale constrained nonlinear programming problems. The natural idea to deal with the large scale structural problem is to somehow decompose the problem into an assembly of small size problems each of which represents an analysis of the behavior of each finite element under a single loading condition. This paper proposes one such way of decomposition based on the duality theory and a recently developed iterative algorithm.

AMS(MOS) Subject Classifications: 90C25, 90C50, 73E20, 73K25

Key Words: Large scale structural analysis, Decomposition Procedure, Plastic limit analysis, Convex programming

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SIGNIFICANCE AND EXPLANATION

Many optimization and design problems arising in structural analysis are formulated and solved as nonlinear programming problems. A variety of formulations and accompanying solution methods have been proposed. Structures with practically meaningful size and complexity tend to give rise to a large number of constraints and variables in the corresponding mathematical model. This tendency is particularly prominent when one uses the conventional Finite Element Method, where a structure is discretized into a large number of small "finite" elements. A major, unresolved difficulty is that the state-of-the-art nonlinear programming theory does not provide a reliable, efficient way of solving a programming problem with a large number of nonlinear constraints. This paper proposes a decomposition procedure for a limit analysis of a certain broad class of (perfectly-plastic) structures. The main benefit of this procedure is that it allows one to solve (many) small-size nonlinear programming problems, each of which corresponds to an individual finite element, instead of having to solve the entire problem involving a potentially huge number of constraints and variables.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

A DECOMPOSITION PROCEDURE FOR LARGE SCALE
OPTIMAL PLASTIC DESIGN PROBLEMS

Ikuyo Kaneko* and Cu Duong Ha**

1. Introduction

In this paper we shall consider an optimal design problem of perfectly plastic structures of potentially large sizes under fairly general assumptions. Using the conventional finite element method the structure is discretized into an assembly of elements, and the problem is formulated as a constrained nonlinear programming problem of finite dimension. A number of ways of formulating the optimal design problem into a mathematical programming problem have been proposed. The unresolved difficulty lies in the facts that (i) the programming problems arising from real-world structural problems tend to have very large sizes and (ii) the state-of-the-art Mathematical Programming theory does not provide reliable and effective algorithms to solve large size constrained nonlinear problems.

Using the finite element method it is inevitable that the number of elements becomes large (hundreds and maybe thousands) if we want to handle engineering structures of practically meaningful size and complexity with reasonable accuracy. In most cases, each finite element is associated with a relatively small number of variables (generalized stresses, design variables, etc.) and constraints (yield conditions, etc.), but because of obvious interactions among elements (e.g., the overall equilibrium condition) we need to consider all the finite elements simultaneously, resulting in a potentially huge number of variables and constraints.

* Department of Industrial Engineering and Department of Computer Sciences, University of Wisconsin-Madison. This author's research is supported, in part, by a grant from the Graduate School, University of Wisconsin-Madison.

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Much research is being done in the field of Large Scale Mathematical Programming but at this point, virtually no reliable, well-tested algorithms are available to solve optimization problems with nonlinear constraints having more than, say, a hundred variables and constraints. It must be pointed out that prospect is much brighter for unconstrained minimization of a differentiable convex function (or maximization of differentiable concave function) especially when the explicit form of derivatives are known. In such a case, some of the most reliable and fast algorithms (such as quasi-Newton methods) can be used to handle large size problems in a reasonably effective manner. Of course, however, the optimal design problems of structures do have nonlinear constraints and thus these unconstrained techniques can not be applied, at least directly.

These considerations lead to the following intuitive idea: Instead of dealing with one large nonlinear program corresponding to the entire structure, can one decompose the problem into small size programs each of which corresponds to the analysis of the behavior of the individual finite elements under a single loading condition and somehow combine the results of these analyses to produce the solution to the entire structure? We believe that this is a natural and effective approach to handle large scale structural optimization problems; and in this paper we shall propose one such approach.

The following is an informal outline of the proposed decomposition procedure (see Section 3 for details). Our basic strategy is to solve the problem by solving its dual. The dual problem is formulated in such a way that it is an unconstrained maximization, with respect to the Lagrange multipliers, of a differentiable concave function with known, explicit derivatives. The function to be maximized in the dual itself

is given as a certain constrained minimization problem (called the augmented primal problem). The augmented primal problem is the same as the original (or primal) problem except that some of the constraints are incorporated to the objective function through the Lagrange multipliers. The dual problem is solved by solving the augmented primal problem repeatedly. This augmented primal problem is decomposed, in two stages, into eventually a collection of small size problems each of which corresponds to a single finite element under a single loading condition.

In the first stage of the decomposition, the augmented primal problem is decomposed into the "sum" of minimization problems each of which is associated with a single finite element under all the loading conditions. In the second stage, each of these problems is partitioned in such a way that the minimization becomes "nested"; i.e. the problem is transformed into an "outer minimization" inside of which many "inner minimizations" are performed. The outer minimization is (virtually) an unconstrained problem with respect to the design variables associated with a single finite element, while each of the inner minimizations is a constrained minimization problem corresponding to a single finite element under a single loading condition given fixed values of the design variables associated with the element. We might note that the dual problem is solved by means of a certain iterative scheme developed recently ([1]-[2]). Figure 1 gives a conceptual outline of our decomposition procedure.

The present work is motivated and inspired by the following two papers: Thierauf [3] and Woo and Schmit [4]. Thierauf proposes an

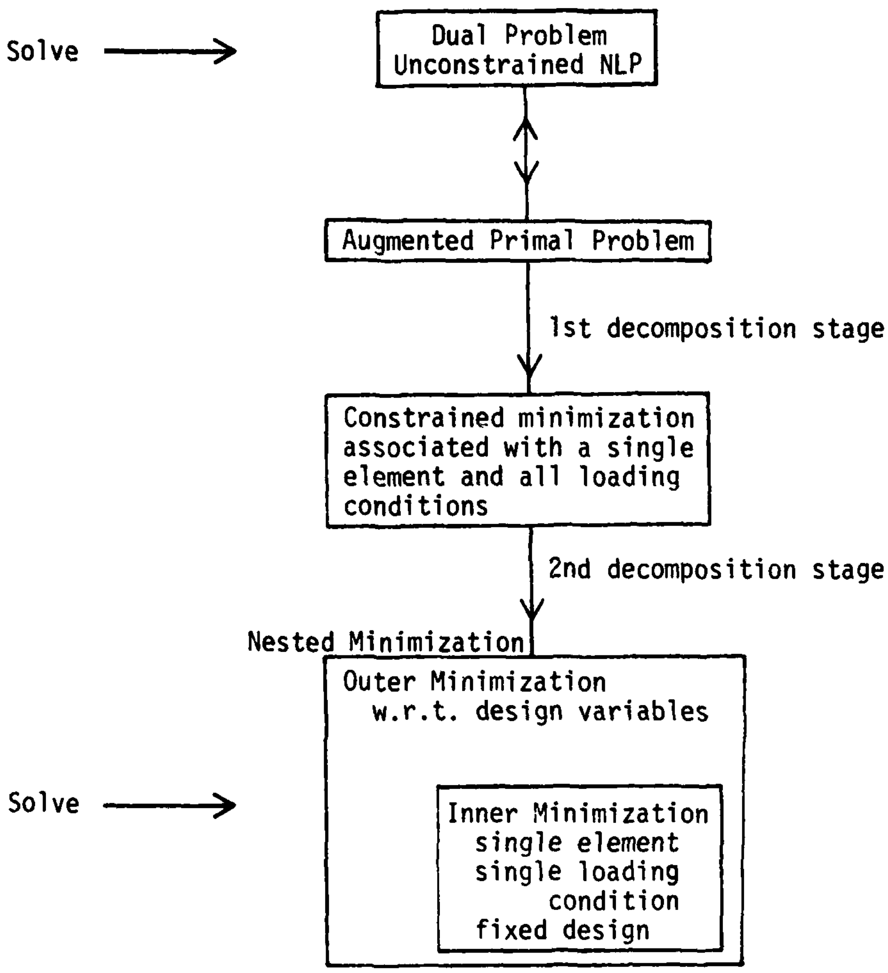

Fig. 1 Conceptual Outline of the Proposed Procedure

iterative scheme for perfectly plastic optimal design problem having a linear objective function and quadratic yield conditions. His procedure has a flavor of decomposition in the sense that in each iteration an elastic problem is solved with respect to <u>all</u> the finite elements but a <u>single</u> loading condition. For a general case with nonlinear objective and yield conditions, Thierauf suggests repeated linear/quadratic approximations in an iterative way.

The decomposition procedure proposed by Woo and Schmit is based on a principle entirely different from ours, but similar to ours in the sense that the problem is decomposed down to a collection of many small size problems corresponding to a single finite element under a single loading condition. The procedure proposed in this present paper seems to have several advantages over that by Woo and Schmit as explained below.

The procedure by Woo and Schmit is an application of Dantzig-Wolf decomposition principle (Dantzig [5]) for (generalized) linear programming. The linearity assumption of the cost function, therefore, is crucial in their method while our procedure only requires convexity and (some) separability. It is well-known ([6]) that the performance of algorithms based on Dantzig-Wolf decomposition principle is highly erratic. Further, the convergence of Woo-Schmit procedure is not guaranteed unless the yield function at each element is piecewise linear. Our procedure is guaranteed to converge under a general convex yield function.

Of course, this doesn't detract the importance of the pioneering work by Woo and Schmit. In this early stage of the obviously important research on the decomposition of large scale structional analysis, all valid approaches must be tried and tested; and our procedure provides one alternative. Another promising approach based on Winkler's factorization/decomposition scheme will be proposed elsewhere by the first author of the present paper in the near future.

The organization of the rest of this paper is as follows. In the next section we shall specify, as a mathematical programming problem, the structural design problem we shall consider in this paper. A description of the proposed decomposition procedure is presented in Section 3. In the fourth and final section we shall discuss some significant simplifications of the proposed procedure when it is applied to a class of certain specific problems; a numerical example and the result of computation will also be given.

2. The Problem

The problem we shall be concerned with can be stated as that of finding a minimal cost design for a perfectly plastic structure subjected to several alternative load conditions under the overall static equilibrium condition and the requirement that no yieldings occur. The corresponding mathematical program looks like:

$$\text{minimize} \quad \sum_{i=1}^{r} w_i(y_i) \tag{1.1}$$

subject to

$$\left.\begin{array}{r} G_i(F_i^j, y_i) \leqq 0 \\ y_i \geqq 0 \\ \sum_{i=1}^{r} N_i F_i^j = P^j \end{array}\right\} \begin{array}{l} i=1,\ldots,r \\ j=1,\ldots,k \\ . \end{array} \quad \begin{array}{r} (1.2) \\ (1.3) \\ (1.4) \end{array}$$

Here, y_i is the vector of design variables associated with the i-th finite element, P^j is the vector representing the j-th loading condition, F_i^j is the vector of (generalized) stresses at the i-th element under the j-th loading condition, $w_i(\cdot)$ is the cost of design y_i, (1.2) represents the yielding conditions and (1.4) denotes the overall static equilibrium.

The framework (1), with the assumptions to be specified later in this section, provides an idealized and simplified, yet fairly broad perfectly plastic optimization model. We note that (1) is in almost exactly the same form as the formulations given in Thierauf [3] and Pape and Thierauf [7], except for some generalities allowed in (1). In the following, we shall explain exactly what these symbols in (1) mean and spell out the set of assumptions we shall make. Our presentation emphasizes idealized mathematical properties and requirements; the (important) question of how valid and useful the formulation (1) is as a structural design model is left to expositions by specialists of structures (such as [3], [7], Pape [8] and Rozvany [9]). In particular, we shall be tacitly making some simpliefied assumptions such as quasistatic loading and "fixed layout".

Consider a perfectly plastic structure which is discretized into r finite elements. The shape and nature of the elements must be chosen depending on the particular problems and are irrelevant to our formulation. Let F_i and y_i denote, respectively, the s_i dimensional vector of (generalized) stresses and the t_i dimensional nonnegative vector of design variables associated with the i-th element, $i=1,\ldots,r$. We assume that for element i there are p_i yield conditions each of which is determined by a differentiable, strictly convex yield function, $i=1,\ldots,r$. Thus, the combined yield condition for the i-th element can be written as

$$G_i(F_i,y_i) \leqq 0, \; i=1,\ldots,r, \tag{2}$$

where G_i is a vector function with p_i components such that each component function is differentiable and strictly convex (jointly) in F_i and y_i. An example of G_i with $p_i = 1$, $t_i = 1$ is

$$G_i(F_i,y_i) = \phi_i(F_i) - y_i,$$

where ϕ_i is a real-valued "plastic potential" function. Figure 2 depicts the yield condition (2) where $p_i = 2$, $t_i = 1$ and $s_i = 2$.

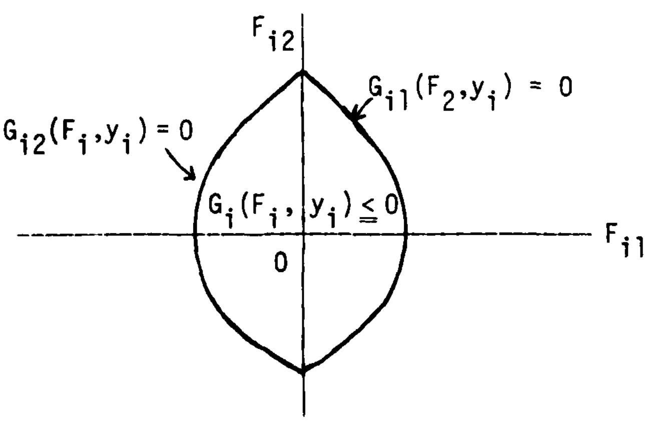

Fig. 2 Piecewise differentiable yield function

The stress vectors $F_1,\ldots,F_r$ for all the elements are required to satisfy the equilibrium condition:

$$\sum_{i=1}^{r} N_i F_i = P, \tag{3}$$

where N_i is an m by s_i matrix, $i=1,\ldots,r$, and P is an m-vector of external nodal loads. We assume that the matrix

$$N = (N_1, N_2, \ldots, N_r)$$

has full row rank.

We let $w(y_1,\ldots,y_r)$ denote the cost (weight, price of material, etc.) of design and assume that $w(\cdot)$ is separable in y_i, i.e.,

$$w(y_1,\ldots,y_n) = \sum_{i=1}^{r} w_i(y_i), \tag{4}$$

where for each $i \in \{1,\ldots,r\}$, $w_i(\cdot)$ is a real-valued, differentiable convex function. Some degree of non-separability can be dealt with in our model but we don't elaborate here. In most practical cases, however, the separability seems a reasonable assumption anyway.

Following Theirauf [3] we formulate the model so that it allows alternative loading conditions. As Thierauf points out, one can design a structure which withstands any convex combination of given k loading conditions by requiring k sets of equilibrium equations. To accomodate this, we let m-vectors $P^1, P^2, \ldots, P^k$ represent the k loading conditions, respectively, and require (in place of (3)):

$$\sum_{i=1}^{r} N_i F_i^j = P^j, \quad i=1,\ldots,r \tag{5}$$

for each $j=1,\ldots,k$, where for each $i \in \{1,\ldots,r\}$, F_i^j is the s_i-vector of stresses associated with element i under the j-th load condition. Accordingly, we replace the yield condition (1) with

$$G_i(F_i^j, y_i) \leq 0,\ j=1,\ldots,k,$$

for each $i \in \{1,\ldots,r\}$. The overall optimization problem is thus given by (1). For notational simplicity we shall use the symbols y and F, respectively, to denote the super vectors,

$$y = (y_1,\ldots,y_r) \text{ and}$$
$$F = (F_1^1,\ldots,F_r^1,F_1^2,\ldots,F_r^2,\ldots,F_1^k,\ldots,F_r^k).$$

Finally, let us note the size of (1). The problem has $\sum_{i=1}^r (t_i + k \cdot s_i)$ variables, $k \cdot \sum_{i=1}^r p_i$ nonlinear and $k \cdot m$ linear constraints (plus $\sum_{i=1}^r t_i$ nonnegativity conditions). Recall that:

r = # of finite elements

m = # of overall equilibrium equations

k = # of loading conditions

t_i = # of design variables for element i

s_i = # of stress componenets for element i; and

p_i = # of yield conditions for element i.

For every i, each of t_i, s_i and p_i is relatively small. Also, k is expected to be relatively small. The number of elements, r, is by far the most crucial quantity in a large scale problem; it could be hundr ds or even thousands.

3. The Decomposition Procedure

To solve the problem (1), we shall consider its dual:

$$\underset{u=(u^1,\dots,u^k)}{\text{maximize}} \quad h(u), \tag{6}$$

where

$$h(u) = \min_{\substack{y_i \geqq 0 \\ G_i(F_i^j, y_i) \leqq 0 \\ i=1,\dots,r,\, j=1,\dots,k}} \left\{ \sum_{i=1}^{r} w_i(y_i) + \sum_{j=1}^{k} (u^j)^T \left(\sum_{i=1}^{k} N_i F_i^j - P^j \right) \right\}. \tag{7}$$

Here, u^j is the m-vector of Lagrange multipliers associated with the equilibrium equations under the j-th loading condition. The (constrained) minimization on the right-hand side of (7) will be referred to as the augmented primal problem. Note that we incorporated only a part of the constraints to the objective function (the Lagrangian) of the augmented primal; this manipulation will play the key role in the decomposition process to be explained below.

If u^* is the maximal solution to (6), and if (y^*,F^*) is the minimal solution to the corresponding augmented primal problem in (7) with $u = u^*$, then the duality theory ensures that (y^*,F^*) is the (global) minimal solution to the original problem (1). For this scheme to be valid, however, we need the assurance that for each given u, the augmented primal problem has a unique optimal solution in (y,F); but this uniqueness is not guaranteed in our problem. To

overcome this difficulty, we shall employ the proximal point algorithm (Rockafellar [1]); i.e., we augment the objective function by adding certain quadratic terms to induce "nice" properties in the augmented primal problem, such as the uniqueness of the solution and the differentiability of h (see below), and solve the augmented problem iteratively, until the true solution is found. Obviously, solving the problem iteratively is costly, but the benefit of having the nice properties far outweighs the added complication.

More specifically, for a given vector $(\bar{y},\bar{F})$, we consider the problem (1) where the objective function (1.1) is replaced with

$$\sum_{i=1}^{r} (w_i(y_i)+\frac{1}{2\lambda}\|y_i-\bar{y}_i\|^2) + \frac{1}{2\lambda}\sum_{i=1}^{r}\sum_{j=1}^{k}\|F_i^j - \bar{F}_i^j\|^2,$$

where λ is some given positive scalar and $\|\cdot\|$ is the Euclidean 2-norm. Thus, instead of the dual problem (6)-(7), we shall consider

$$\underset{u=(u^1,\ldots,u^k)}{\text{maximize}} \quad h(u) \tag{8}$$

with

$$h(u) = \min_{\substack{y_i \geq 0 \\ G_i(F_i^j,y_i) \leq 0 \\ i=1,\ldots,r,j=1,\ldots,k}} \{\sum_{i=1}^{r}(f_i(y_i)+\sum_{j=1}^{k}g_{ij}(F_i^j))+\sum_{j=1}^{k}(u^j)^T(\sum_{i=1}^{r}N_iF_i^j-P^j)\} \tag{9}$$

where we set

$$f_i(y_i) = w_i(y_i) + \frac{1}{2\lambda} \|y_i - \bar{y}_i\|^2, \text{ and} \tag{10.1}$$

$$g_{ij}(F_i^j) = \frac{1}{2\lambda} \|F_i^j - \bar{F}_i^j\|^2, \tag{10.2}$$

$$i=1,\ldots,r, \ j=1,\ldots,k.$$

It must be noted that f_i and g_{ij} are functions of $\bar{y}_i$ and $\bar{F}_i^j$ as well as y_i and F_i^j, respectively, and that in fact $\bar{y}$ and $\bar{F}$ are changed from one iteration to another; but we suppress them for notational simplicity.

Let u^* be the dual optimum and let (y^*,F^*) be the corresponding minimizer in (9). If (y^*,F^*) coincides with $(\bar{y},\bar{F})$, then it can be shown (compare (6)-(7) and (8)-(9)) that (y^*,F^*) is the optimal solution to the original problem (1).† Otherwise, we replace $(\bar{y},\bar{F})$ with (y^*,F^*) and repeat the process. This process is guaranteed to converge, with a reasonably good convergence rate†† (see [2]) and computational results (see Ha [10]) have been encouraging. This iterative process is the basic iteration scheme providing the "highest echelon" loop in our decomposition procedure. The augmented primal problem in (9) will be decomposed, in two stages, into eventually small size problems each of which is associated with a single structural element under a single loading condition; but we shall first examine the properties of the augmented primal further.

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† Of course, in practice, we accept a solution (y^*,F^*) if it is "sufficiently" close to $(\bar{y},\bar{F})$.

†† If $w_i(y_i)$ is linear for each i, then the process terminates after a finite number of iterations.

For each u, let $(y(u), F(u))$ denote the unique solution of the augmented primal problem. It can be shown (see [11]) that $h(u)$ is strictly concave and differentiable in u. Furthermore, the derivatives of $h(u)$ is known explicitly; i.e., the gradient vector of $h(u)$ with respect to u^j is given by

$$\nabla_{u^j} h(u) = \sum_{i=1}^{r} N_i F_i^j(u) - P^j,$$

where $F_i^j(u)$ is the appropriate component of $F(u)$ defined above. We note that all these crucial and nice properties are the results of adding the quadratic terms. The maximization problem (8) does have a large size (the dimension of u is $k \cdot m$); however, the facts that $h(u)$ is differentiable, strictly concave with known, explicit derivatives and that the problem is unconstrained ensure that the optimization can be performed effectively by using some of the most reliable and fast unconstrained nonlinear programming routines (such as a quasi-Newton method).

The first stage of the decomposition process is done by observing that the optimization in (8) can be accomplished by solving the following subproblem separately for each i:

$$\text{minimize} \quad f_i(y_i) + \sum_{j=1}^{k} (g_{ij}(F_i^j) + (u^j)^T N_i F_i^j) \qquad (11.1)$$

$$\text{subject to} \quad G_i(F_i^j, y_i) \leqq 0, \; j=1,\ldots,k \qquad (11.2)$$

$$y_i \geq 0. \qquad (11.3)$$

This is easy to see if we rearrange some terms in (9) and rewrite it as

$$h(u) = \min_{\substack{y_i \geq 0 \\ G_i(F_i^j, y_i) \leq 0 \\ i=1,\ldots,r,\, j=1,\ldots,k}} \{\sum_{i=1}^{r}(f_i(y_i)+\sum_{j=1}^{k}(g_{ij}(F_i^j))+\sum_{i=1}^{r}\sum_{j=1}^{k}(u^j)^T N_i F_i^j - \sum_{j=1}^{k}(u^j)^T P^j\}$$

$$= -\sum_{j=1}^{k}(u^j)^T P^j + \sum_{i=1}^{r} \min_{\substack{y_i \geq 0 \\ G_i(F_i^j, y_i) \leq 0 \\ j=1,\ldots,k}} \{f_i(y_i)+\sum_{j=1}^{k}(g_{ij}(F_i^j)+(u^j)^T N_i F_i^j)\}.$$

Note that the problem (11) corresponds to a <u>single</u> finite element and <u>all</u> k loading conditions.

The second stage of the decomposition is obtained by re-expressing the problem (11) as the following nested minimization (it is not difficult to verify the validity of this "partitioning"):

$$\underset{y_i \geq 0}{\text{minimize}} \; \{f_i(y_i)+\sum_{j=1}^{k} v_{ij}(y_i)\} \tag{12}$$

where

$$v_{ij}(y_i) = \min_{G_i(F_i^j, y_i) \leq 0} \{g_{ij}(F_i^j)+(u^j)^T N_i F_i^j\}. \tag{13}$$

Namely, (11) is solved by performing the <u>outer minimization</u> (12) with respect to y_i, where the value of each $v_{ij}(y_i)$ is given by solving the following <u>inner minimization</u>:

$$\text{minimize} \quad g_{ij}(F_i^j) + (u^j)^T N_i F_i^j \tag{14.1}$$

$$\text{subject to} \quad G_i(F_i^j, y_i) \leq 0. \tag{14.2}$$

It can be shown (see Rockafellar [11]) that the function

$$f_i(y_i) + \sum_{j=1}^{k} v_{ij}(y_i)$$

is convex in y_i. Further, under certain mild conditions (see Ha [10]) this function is differentiable in y_i and an explicit form its derivatives can be obtained. These, as well as the fact that the dimension of y_i is small, lead to an efficient minimization (using a quasi-Newton algorithm, for example).

Now, the inner minimization, (14), is a (relatively) small size constrained optimization problem with s_i variables and p_i constraints, which corresponds to a single finite element under a single loading condition for given values of the design variables. By hypothesis, $G_i(F_i^j, y_i)$ is differentiable and convex in F_i^j (for fixed y_i); the objective function is quadratic (c.f. (10)). These properties and the smallness of the problem ensures an efficient solution of the inner minimization.

To summarize the proposed procedure, we look at it as a "modular computer package" consisting of several subroutines as follows.

Preparation:

Choose an initial point $(\bar{y}, \bar{F})$. Add the quadratic terms to the objective function (as in (10)).

Main Routine (Basic Iteration Scheme)

Call subroutine MAXH to solve the problem (8)-(9). Let (y^*,F^*) be the optimal solution to the augmented primal problem (9) corresponding to the optimal solution u^* of (8). If $\|(y^*,F^*)-(\bar{y},\bar{F})\| \leq \varepsilon$ for a prescribed tolerance $\varepsilon > 0$, then stop ((y^*,F^*) is optimal); otherwise, set $\bar{y} =: y^*$, $\bar{F} =: F^*$ and repeat.

Subroutine MAXH (Input: $(\bar{y},\bar{F})$ Output: (y^*,F^*))

Use a quasi-Newton algorithm to solve (8)-(9); call subroutine EVAHDH to evaluate $h(u)$ and its derivatives needed for the algorithm.

Subroutine EVAHDH (Input: $(\bar{y},\bar{F})$, u Output: $h(u)$, $\nabla h(u)$)

Call subroutine OUTMIN to solve (12)-(13) for each $i=1,\ldots,r$; let $(y_i(u);\ F_i^1(u),\ F_i^2(u),\ldots,F_i^k(u))$ be the solution, $i=1,\ldots,r$. The values of $h(u)$ and its derivatives are given, respectively, by:

$$h(u) = \sum_{i=1}^{r}(f_i(y_i(u)) + \sum_{j=1}^{k} g_{ij}(F_i^j(u))) + \sum_{j=1}^{k}(u^j)^T(\sum_{i=1}^{r} N_i F_i^j(u) - P^j); \text{ and}$$

$$\nabla_{u^j} h(u) = \sum_{i=1}^{r} N_i F_i^j(u) - P^j,\ j=1,\ldots,k.$$

Subroutine OUTMIN (Input: $(\bar{y},\bar{F})$, u, i Output: $(y(u), F(u))$)

Solve the problem (12) for y_i; call subroutine INMIN to evaluate $v_{ij}(y_i)$, $j=1,\ldots,k$.

<u>Subroutine INMIN</u> (Input: $\bar{F}$, u, i, y_i, j Output: $v_{ij}(y_i)$)

Solve the problem (14).

Figure 3 shows the relationships among these subroutines.

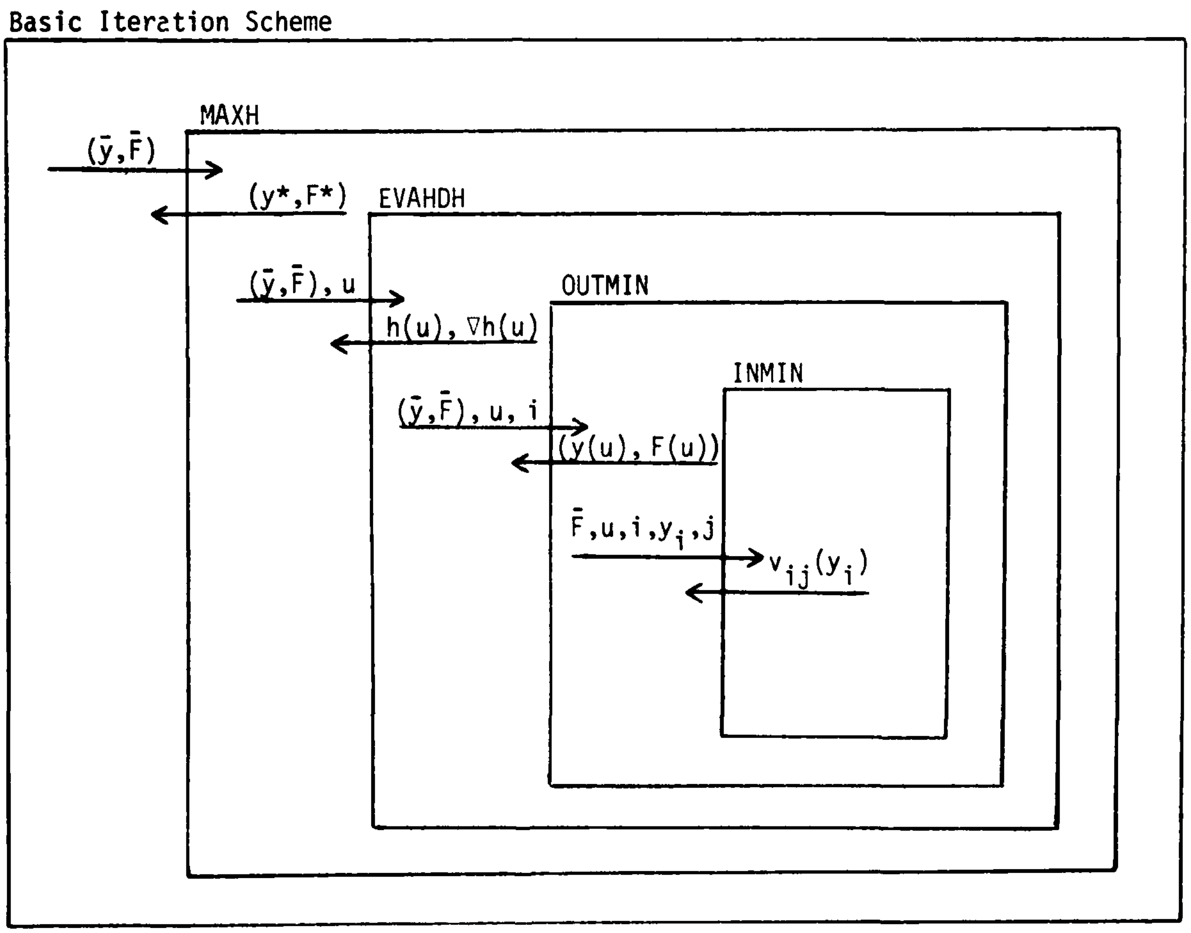

Fig. 3 Summary of the Proposed Procedure

4. A Special Case and an Example

The decomposition procedure was presented in the previous section for a fairly broad class of problems. Most often, the particular problem under consideration has many special features which lead to some simplifications of the procedure. In this section we shall examine one such case.

The problem we shall consider here is given by:

$$\text{minimize} \quad \sum_{i=1}^{r} w_i(y_i) \tag{15.1}$$

$$\left.\begin{array}{ll} \text{subject to} & \frac{1}{2}(F_i^j)^T D_i F_i^j \leqq y_i \qquad (15.2) \\ & y_i \geqq 0 \qquad (15.3) \\ & \sum_{i=1}^{r} N_i F_i^j = P^j \qquad (15.4) \end{array}\right\} \begin{array}{l} i=1,\dots,r \\ j=1,\dots,k \\ , \end{array}$$

where D_i is an s_i by s_i symmetric positive definite matrix for each $i \in \{1,\dots,r\}$. This is the special case of the general problem (1), where $t_i = p_i = 1$ for each i and G_i is the real-valued function with

$$G_i(F_i^j, y_i) = \frac{1}{2}(F_i^j)^T D_i F_i^j - y_i, \; i=1,\dots,r.$$

Namely, each finite element has a single design variable and a single, quadratic yield condition. In this case, rather significant simplifications can be made in the proposed decomposition procedure particularly with respect to solving (12)-(13).

Consider the nested minimization problem (12)-(13) (here, u is given). Since y_i is a scalar, the outer minimization is a one-dimensional optimization, or a "line search". It can be shown (not difficult) that the function to be minimized is differentiable and convex where the derivative is equal to the value of optimal Lagrange multiplier. Thus, the outer minimization can be performed extremely efficiently by using a one-dimensional quasi-Newton algorithm. By using other properties of $v_{ij}(\cdot)$, further simplifications are possible (but not elaborated here). It turns out, also, that the inner minimization to evaluate $v_{ij}(y_i)$ reduces to solving one (nonlinear) equation with one unknown and so it can be performed in an extremely simple manner. Details are described at the end of this section.

We wrote a Fortran computer code to solve an optimal design problem of the type (15) using the proposed decomposition procedure and the simplifications in solving (12)-(13) outlined above. We applied the algorithm to solve a numerical example treated in Thierauf [3] (Example 1 on p. 146) and it computed an optimal solution after 4 "outer" iterations(Basic Iteration Scheme) and 38 quasi-Newton steps (75 function evaluations). The solution is feasible up to the error of magnitude 10^{-4} and its objective value is slightly lower than that given in Thierauf's paper. We have solved a few other, larger size problems using the algorithm also.

In the remainder of this section we shall explain how to solve the problem (14) efficiently when the problem is given by (15). For notational simplicity we shall write the problem as follows:

$$\text{minimize} \quad \tfrac{1}{2}x^T x - c^T x \tag{16.1}$$

$$\text{subject to} \quad \tfrac{1}{2}x^T D x \leqq \alpha, \tag{16.2}$$

where x is an s-vector variable, c is a constant s-vector, D is a constant symmetric and positive define matrix and α is a given positive scalar. Clearly the inner minimization problem (14) for (15) is obtained by setting

$$x = F_i^j,\ D = D_i,\ \alpha = y_i,$$

$$c = \frac{1}{2\lambda}\bar{F}_i^j - (u^i)^T N_i,$$

and by adding the constant $\frac{1}{2\lambda}(\bar{F}_i^j)^T \bar{F}_i^j$ to the objective of (16).

The problem is obviously feasible and is a convex program with a strictly convex objective function; thus it has a unique minimum, which is characterized by the Kuhn-Tucker conditions:

$$x - c + \xi Dx = 0 \tag{17.1}$$

$$\xi \geqq 0,\ x^T Dx \leqq \alpha \tag{17.2}$$

$$\xi \cdot (\alpha - x^T Dx) = 0, \tag{17.3}$$

where ξ is the scalar Lagrange multiplier. Clearly, the optimal Lagrange multiplier is zero if and only if $c^T Dc \leqq a$; if $\xi = 0$, then $x = c$ is the optimal solution to (16). Thus, the first step of the procedure to solve (16) should be to check if $c^T Dc \leqq \alpha$ holds (if so $x = c$ is the optimum).

Suppose, now, that $x = c$ is not feasible in (16) and so the optimal Lagrange multiplier is strictly positive. To solve (16) under this circumstance, we take the dual approach as we did for the original problem (1); the dual of (16) is given by

$$\underset{\xi \geqq 0}{\text{maximize}} \quad a(\xi), \tag{18}$$

where

$$a(\xi) = \min_{x} \{\tfrac{1}{2}x^T x - c^T x + \tfrac{1}{2}\xi x^T Dx\} - \xi\alpha. \tag{19}$$

For each fixed ξ, let $x(\xi)$ denote the minimizer in (19). Then it is not difficult to see that

$$x(\xi) = (I+\xi D)^{-1}c, \tag{20}$$

where I is the s by s identity. Hence we have

$$\begin{aligned} a(\xi) &= \tfrac{1}{2}x(\xi)^T x(\xi) - c^T x(\xi) + \xi x(\xi)^T Dx(\xi) - \xi\alpha \\ &= \tfrac{1}{2}x(\xi)^T[x(\xi) - c + \xi Dx(\xi)] - \tfrac{1}{2}c^T x(\xi) - \xi\alpha \\ &= -\tfrac{1}{2}c^T x(\xi) - \xi\alpha \\ &= -\tfrac{1}{2}c^T(I+\xi D)^{-1}c - \xi\alpha. \end{aligned}$$

As before, $a(\xi)$ is differentiable concave in ξ and so the dual problem (18) can be solved by obtaining a nonnegative (actually positive) solution ξ to the single equation:

$$\frac{d}{d\xi}[\tfrac{1}{2}c^T(I+\xi D)^{-1}c+\xi\alpha] = 0,$$

or

$$\frac{d}{d\xi}c^T(I+\xi D)^{-1}c = -2\alpha. \tag{21}$$

For instance, if $s = 1$, then the equation (21) becomes

$$\frac{d}{d\xi} \frac{c^2}{(1+\xi D)} = -2\alpha$$

where both c and D are scalars and thus the positive root of equation is given by[†]

$$\xi = -\frac{1}{D} + \frac{c}{\sqrt{2\alpha D}} .$$

In a general case an alternative form of the equation (21) may be obtained as follows. Let $\{\eta_i : i=1,\ldots,s\}$ and $\{h^i : i=1,\ldots,s\}$ be the sets of eigenvalues and eigenvectors, respectively, of the matrix D. By the assumptions on D, η_i is a positive scalar and h^i is real, for each i. It is not difficult to show that h^i is also an eigenvector of the matric $(I+\xi D)^{-1}$ with the associated eigenvalue $1/(1+\xi\eta_i)$ for each i. Let $\mu = (\mu_1,\ldots,\mu_s)^T$ be the unique solution of the system

$$c = \textstyle\sum_{i=1}^{s} \mu_i h^i; \quad \text{or}$$

$$\mu = [h^1,\ldots,h^s]^{-1} c. \tag{22}$$

Then it can be shown (not difficult) that

$$x(\xi) = (I+\xi D)^{-1} c = \sum_{i=1}^{s} \left(\frac{\mu_i}{1+\xi\eta_i}\right) h^i .$$

[†] This ξ is positive if and only if $c^2D/2 > \alpha$; if $c^2D/2 \leq \alpha$, or $cDc/2 \leq \alpha$, then $x = c$ is feasible and thus the problem would be solved by the "first step", i.e. the check of feasibility of $x = c$.

Therefore, the equation (21) becomes

$$\sum_{i=1}^{s} (c^T h^i) \left[\frac{\eta_i \mu_i}{(1+\xi\eta_i)^2} \right] = 2\alpha .$$

This process requires the determination of all eigenvalues and eigenvectors of D and the matrix inversion necessary in (22). Most computing centers are equipped with computer packages to do these operations.

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A decomposition procedure is proposed in this paper for solving a class of large scale optimal design problems for perfectly plastic structures under several alternative loading conditions. The conventional finite element method is used to cast the problem into a finite dimensional constrained nonlinear programming problem. Structures of practically meaningful size and complexity tend to give rise to a large number of variables and constraints in the corresponding mathematical model. The difficulty is that the state-of-the-art

(continued)

20. ABSTRACT (Cont'd)

Mathematical Programming theory does not provide reliable and efficient ways of solving large scale constrained nonlinear programming problems. The natural idea to deal with the large scale structural problem is to somehow decompose the problem into an assembly of small size problems each of which represents an analysis of the behavior of each finite element under a single loading condition. This paper proposes one such way of decomposition based on the duality theory and a recently developed iterative algorithm.

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